Conformal invariance and QCD Pomeron vertices in the $1/N_c$ limit

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Abstract

Using the dipole framework for QCD at small x_{Bj} in the $1/N_c$ limit, we derive the expression of the $1 \to p$ dipole multiplicity density in momentum space. This gives an analytical expression for the $1 \to p$ QCD Pomeron amplitudes in terms of one-loop integration of effective vertices in transverse momentum. Conformal invariance and a Hilbert space construction for dipole correlation functions are the main tools of the derivation. Relations with conformal field theories in the classical limit are discussed.

1 Introduction

QCD being a scale-invariant field theory at classical level, it is natural to discuss whether scaling symmetries can be exactly or approximately preserved in some kinematical regime. For instance, in high-energy reactions, scale invariance in the transverse plane or its extension to 2-dimensional conformal invariance have been discussed. The existence of such symmetries could give new insights into the properties of the theory. However, the absence of a solution of QCD at strong coupling leaves the discussion open concerning

the non-perturbative aspects of the problem. The interest of resummation of perturbative QCD at high energy (small Bjorken x_{Bj}) is to allow a theoretical investigation of high-energy amplitudes in a regime which can be computed from our present knowledge of the theory.

The main outcome of the perturbative resummation calculations at leading $\log x_{Bj}$ in the relevant domain is the emergence of a Regge singularity, the so-called BFKL Pomeron, which is a compound state of 2 reggeized gluons [1]. In impact parameter space, the BFKL Pomeron contributions can be expressed [2] in terms of the formation and interactions of colorless $q\bar{q}$ dipoles in the $\frac{1}{N_c}$ limit of small x_{Bj}-QCD. Interestingly enough, the calculation of various processes in both BFKL and QCD dipole picture frameworks exhibits properties related to scale and conformal invariance.

First of all, it was shown [3] that the BFKL equation possesses a basic global $SL(2,\mathbb{C})$ conformal invariance which is an useful technical tool. Indeed, conformal invariance tools appear in other calculations related to the BFKL Pomerons, for instance the $2 \rightarrow 4$ gluon amplitudes [4]. Using the same invariance it is possible to derive [5] an equivalence between s-channel dipole-dipole interactions and the t-channel BFKL Pomeron.

These conformal invariance properties, and the possible extension to conformal field theories (CFT) have been recently pointed out in relation with studies on multi-pomeron (multi-dipole) processes. The solution of the equations for the $1 \rightarrow p$ dipole multiplicity density in impact parameter space is found to be identical to Shapiro-Virasoro closed string amplitudes [6]. In fact, the result of reference [6] goes beyond $SL(2,\mathbb{C})$ invariance since it relies on the specific form of the BFKL kernel. It was proposed [7] that the interaction vertices of three BFKL Pomerons can be related to correlation functions in two-dimensional conformal field theory (CFT), satisfying conformal bootstrap relations. It was checked that this leads to the same amplitudes as found in the dipole formulation.

In fact, the direct calculation from QCD Feynman diagrams is only presently available for the $2 \rightarrow 4$ gluon amplitudes [4]. It can easily be checked that, when considering three colour-singlet gluon-gluon (e.g. QCD Pomeron) channels in the $1/N_c$ limit, it leads to the expression obtained in the dipole formulation. No direct calculation of amplitudes with more channels are yet available.

Our goal in the present paper is to investigate the general $1 \rightarrow p$ QCD Pomeron amplitudes using relations between BFKL-dipole amplitudes and correlation functions of suitably defined two-dimensional operators. The

main tool of our derivations is to use a Hilbert space construction of the operators representing the dipole physical states and their correlation functions. Our main observation is that a formulation of the correlation functions can be given in the momentum space representation leading to a simple and physically appealing expression of the QCD Pomeron amplitudes. This representation is realized in terms of one-loop amplitudes built from explicit effective Pomeron vertices.

The plan of our paper is the following. In section **2** we introduce a Hilbert space construction of dipole correlation functions in impact parameter space. In section **3** we derive the corresponding momentum space representation which, in the following part **4**, is shown to possess a simple algebraic and geometrical formulation. In **5**, we discuss the CFT interpretations of our results and give our conclusions in section **6**. Two appendices complete the paper.

2 Hilbert space construction of dipole correlation functions

The $1\rightarrow p$ dipole multiplicity density (i.e. the probability for finding p dipoles $\rho_{b_0}\rho_{b_1},\ldots,\rho_{p_0}\rho_{p_1}$ in an initial one $\rho_{a_0}\rho_{a_1}$) is obtained [6] from the solution of a integro-differential equation[1]. It reads:

$$
\begin{aligned}
d_p(\rho_{a_0}\rho_{a_1},\ldots,\rho_{p_0}\rho_{p_1}) &= \int dh_0 dh_1 \ldots dh_p \ \frac{1}{2a_{h_0}\ldots a_{h_p}} \ \times \\
&\times \frac{1}{\omega-\omega_{h_0}} \ \frac{1}{\omega_{h_1}+\ldots+\omega_{h_p}-\omega} \ \frac{1}{|\rho_{a_0a_1}\ldots\rho_{p_0p_1}|^2} \times \\
\times \int d^2\rho_\alpha d^2\rho_\beta \ldots d^2\rho_\pi \ &\bar{E}^{h_0}(\rho_{a_0\alpha},\rho_{a_1\alpha})\ldots\bar{E}^{h_p}(\rho_{0\pi},\rho_{1\pi}) \ B_{1\rightarrow p},
\end{aligned} \tag{1}
$$

where $a_h = \frac{\pi^4/2}{\nu^2+n^2/4}$ and

$$
B_{1\rightarrow p} = \int \frac{d^2\rho_0\ldots d^2\rho_p}{|\rho_{01}\ \rho_{12}\ldots\rho_{p0}|^2} \ E^{h_0}(\rho_{0\alpha},\rho_{1\alpha})E^{h_1}\left(\rho_{1\beta},\rho_{2\beta}\right)\ldots E^{h_p}\left(\rho_{p\pi},\rho_{0\pi}\right) \ , \tag{2}
$$

[1]The multiplicity density is normalized by the unit of initial dipole size squared $d_p \equiv n_p/\rho^2_{a_0a_1}$, where n_p is the quantity computed in [6].

with $\rho_{ij} = \rho_i - \rho_j$ (resp. $\bar{\rho}_{ij} = \bar{\rho}_i - \bar{\rho}_j$) . In formula (2),

$$E^h(\rho_{i\delta}, \rho_{j\delta}) = (-1)^n \left(\frac{\rho_{ij}}{\rho_{i\delta}\rho_{j\delta}}\right)^h \left(\frac{\bar{\rho}_{ij}}{\bar{\rho}_{i\delta}\bar{\rho}_{j\delta}}\right)^{\tilde{h}} \tag{3}$$

are [3] the $SL(2, \mathbb{C})$ eigenvectors labeled by the quantum numbers of the irreducible unitary representations, namely $h = i\nu + \frac{1-n}{2}$, $\tilde{h} = 1 - \bar{h} = i\nu + \frac{1+n}{2}$, ($n \in \mathbb{Z}$, $\nu \in \mathbb{R}$). The variable ω in formula (1) is the Mellin-transform conjugate of rapidity while the functions $\omega_{h_0}, ..., \omega_{h_p}$ are the BFKL eigenvalues associated with the $p + 1$ dipoles involved in the process, i.e.

$$\omega_h = \frac{2\alpha N_c}{\pi} \Re e \left\{ \psi(1) - \psi\left(\frac{1+n}{2} + i\nu\right) \right\}. \tag{4}$$

In the following, we shall consider the expression of the functions $B_{1\to p}$ as correlation functions, namely

$$B_{1\to p} \equiv \left\langle 0 | \Phi^{h_0}(\rho_\alpha) \Phi^{h_1}(\rho_\beta) ... \Phi^{h_p}(\rho_\pi) | 0 \right\rangle \tag{5}$$

where the $\Phi^h(x)$ are suitably defined operators. Such a formulation has been discussed in ref. [3] in comparison with the general $SL(2, \mathbb{C})$ invariant Green functions [8]. Along these lines, an approach [7] considered correlation functions of quasi-primary operators [9], where 4-point correlation functions obey conformal bootstrap constraints. in the following we shall derive an explicit construction of the relevant operators in a different context.

An inspiring example for our derivation is the expression of correlation functions in the mini-superspace (classical) limit [10, 11] of the coset $H_3^+ \equiv SL(2, \mathbb{C})/SU(2)$ Wess-Zumino-Novikov-Witten (WZNW) conformal field theory [12]. In this case, the derivation is based on an explicit construction of the primary fields (operators) $\Psi^j(x)$, where j labels some $SL(2, \mathbb{C})$ representation and $x, \bar{x}$ are auxiliary variables, that is, not belonging to the 2-dimensional coordinate space on which the conformal fields are defined. In fact these auxiliary variables span the set of states in a given infinite-dimensional representation. The operators act on the Hilbert space $L^2(H_3^+)$ by multiplication by functions forming an (overcomplete) basis of the Hilbert space. The correlation functions are realized as the expectation value of a product of these operators evaluated in a state defining the $SL(2, \mathbb{C})$ invariant vacuum.

Starting with our derivation, the following ingredients are defined in order to give an operator realization of the correlation functions (5):

i) The Hilbert space is choosen to be the set of functions $f \in \mathcal{H} = L^2(\mathbb{C}^2)$, with a representation of the action of the $SL(2,\mathbb{C})$ group given by

$$[T(g)f](\rho_1,\rho_2) = (c\rho_1+d)^{-1}(c\rho_2+d)^{-1} \times \{a.h.\}\ f(\rho_1',\rho_2') \tag{6}$$

where $\rho' = \frac{a\rho+b}{c\rho+d}$, and the shortened notation $\times\{a.h.\}$ is used for multiplying by the antiholomorphic counterpart. The $SL(2,\mathbb{C})$ matrix g is by definition

$$g = \left(\begin{array}{cc} a & c \\ b & d \end{array} \right) . \tag{7}$$

ii) The $SL(2,\mathbb{C})$ invariant vacuum vector is $|0\rangle = \delta(\rho_1 - \rho_2)$. Indeed, it is easy to check that, using (6), it verifies the relation $T(g)\,|0\rangle = |0\rangle$.

iii) The operators $\Phi^h(x)$ act by

$$[\Phi^h(x)f](\rho,\rho'') = \int \frac{d^2\rho'}{|\rho'-\rho|^2} E^h(\rho - x, \rho' - x) f(\rho',\rho'') . \tag{8}$$

To check that we indeed recover the correct expression (5) we see that acting with the chain of operators $\Phi^{h_0}(\rho_\alpha)\Phi^{h_1}(\rho_\beta)\ ...\Phi^{h_p}(\rho_\pi)$ on the vacuum vector $|0\rangle$ gives the following function of the variables ρ_0 and ρ_{p+2}

$$\int \frac{d^2\rho_1 ... d^2\rho_{p+1}}{|\rho_{01}...\rho_{p(p+1)}|^2} E^{h_0}(\rho_{0\alpha},\rho_{1\alpha}) ... E^{h_p}(\rho_{p\pi},\rho_{(p+1)\pi})\ \delta(\rho_{p+1} - \rho_{p+2}) . \tag{9}$$

Now taking the scalar product with the vacuum vector amounts to multiplying the above expression by $\delta(\rho_0 - \rho_{p+2})$ and integrating over ρ_0 and ρ_{p+2}. We then recover the dipole formula (2).

The expression (8) enables us to exhibit the conformal properties of the dipole amplitudes. One shows that

$$T(g)\Phi^h(x')T(g)^{-1} = (cx+d)^{2h} \times \{a.h.\}\ \Phi^h(x) \tag{10}$$

where $x' = \frac{ax+b}{cx+d}$.

In the following we will go to momentum space and find a compact realization, which exhibits remarkably simple properties with respect to $SL(2,\mathbb{C})$ transformations of momenta. Also it has a direct physical significance as giving an expression for the $1 \to p$ multiplicity density of dipoles in the momentum representation, and thus the $1 \to p$ QCD Pomeron amplitudes in the $1/N_c$ limit.

3 Correlation functions in momentum space

The action of the operators $\Phi^h(x)$ defined in (8) in the Fourier transformed space of functions

$$\tilde{f}(k,k') = \frac{1}{(2\pi)^4}\int d^2\rho d^2\rho' \; e^{i(k\rho+k'\rho')} f(\rho,\rho'), \tag{11}$$

is given by:

$$[\Phi^h(x)f](k,k'') = \int d^2k' \int d^2\rho d^2\rho' \; e^{ik\rho - ik'\rho'} \frac{E^h(\rho-x,\rho'-x)}{|\rho-\rho'|^2}\tilde{f}(k',k'') \; . \tag{12}$$

A substantial simplification occurs when we will consider the Fourier transform of the Φ^h operator with respect to x:

$$\Phi^h(q) = \int d^2x \; e^{-iqx}\Phi^h(x) \; . \tag{13}$$

Now the vacuum becomes $|0\rangle = \delta(k+k')$ and the action of the operators becomes just a multiplication by a function combined with a shift in the argument:

$$[\Phi^h(q)\tilde{f}](k,k') = \mathcal{E}^h_{k,q}\cdot \tilde{f}(k-q,k') \; , \tag{14}$$

where we have defined

$$\mathcal{E}^h_{k,q} \equiv \int d^2\rho d^2\rho' \; e^{iq\rho' + ik(\rho-\rho')} \; \frac{E^h(\rho,\rho')}{|\rho-\rho'|^2} \; . \tag{15}$$

Since the rôle of the second coordinate is to generate a delta function for the conservation of momentum, we may redefine the Hilbert space, vacuum state and operators in such a way that:

$$\left\langle \Phi^{h_0}(q_0)\ldots\Phi^{h_p}(q_p)\right\rangle = \delta(q_0+\ldots+q_p) \; \left\langle \phi^{h_0}(q_0)\ldots\phi^{h_p}(q_p)\right\rangle \; , \tag{16}$$

where the $\phi^h(q)$ act on the Hilbert space $L^2(\mathbb{C})$ with the vacuum $|0\rangle = 1$ by

$$[\phi^h(q)f](k) = \mathcal{E}^h_{k,q}\cdot f(k-q) \tag{17}$$

The result of the Fourier transform is

$$\begin{aligned} A^{h_0,\ldots,h_p}(q_0,\ldots,q_p) \; &\equiv \; \left\langle \phi^{h_0}(q_0)\ldots\phi^{h_p}(q_p)\right\rangle \\ &= \; \int d^2k \; \mathcal{E}^{h_0}_{k,q_0}\mathcal{E}^{h_1}_{k-q_0,q_1}\ldots\mathcal{E}^{h_p}_{k-q_0-\ldots-q_{p-1},q_p} \; . \end{aligned} \tag{18}$$

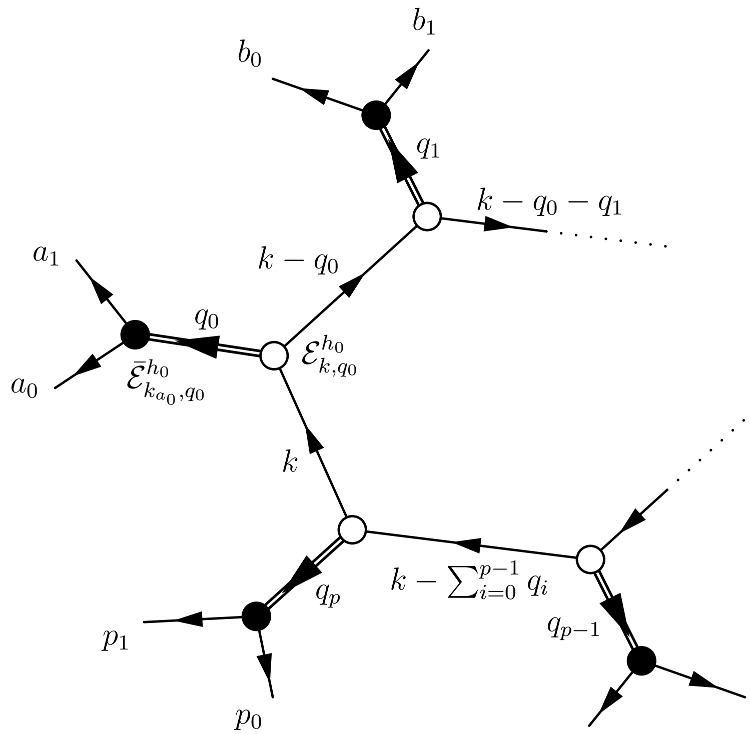

Figure 1: *Graphical representation of the* $1 \to p$ *QCD Pomeron amplitude. White circles: internal* $\mathcal{E}^h_{k,q}$ *vertex functions entering the one-loop integral; Black circles: external (complex conjugate) vertices* $\bar{\mathcal{E}}^h_{k,q}$ *coupling the external gluons to the interacting BFKL Pomerons; Double lines: BFKL Pomerons.*

Thus, all dipole correlators are just expressed by a *single* integral over a product of $\mathcal{E}^h_{k,q}$ functions.

Using our Fourier transformed result (18),

$$B_{1\to p} = \int d^2q_0...d^2q_p \; \delta(q_0 + ... + q_p) \; e^{i(q_0\rho_\alpha + ... + q_p\rho_\pi)} \; A^{h_0,...,h_p}(q_0,...,q_p) \; , \quad (19)$$

together with the one for the functions $\bar{\mathcal{E}}^h_{k,q}$, (see (15)) namely

$$\frac{\bar{E}^h(\rho,\rho')}{|\rho-\rho'|^2} \equiv \frac{1}{16\pi^4} \int d^2k d^2k' \; e^{ik'\rho' + ik(\rho-\rho')} \; \bar{\mathcal{E}}^h_{k,k'} \; , \quad (20)$$

and inserting them in formula (1), we get

$$\mathcal{A}(k_{a_0}k_{a_1},...,k_{p_0}k_{p_1}) \quad \equiv \quad \int \frac{d^2\rho_{a_0}}{2\pi} ... \frac{d^2\rho_{p_1}}{2\pi} \; e^{-i(\rho_{a_0}k_{a_0} + ... + \rho_{p_1}k_{p_1})} d_p(\rho_{a_0}...\rho_{p_1})$$

$$
\begin{aligned}
&= \int dh_0 dh_1 ... dh_p \, \frac{1}{2a_{h_0}...a_{h_p}} \times \\
&\times \frac{1}{\omega - \omega_{h_0}} \frac{1}{\omega_{h_1} + ... + \omega_{h_p} - \omega} \times \mathcal{G} \ ,
\end{aligned}
\tag{21}
$$

with

$$
\mathcal{G} \equiv \delta(q_0 + ... + q_p) \ A^{h_0,...,h_p}(q_0, ..., q_p) \ \bar{\mathcal{E}}^{h_0}_{k_{a_0},q_0} ... \bar{\mathcal{E}}^{h_p}_{k_{p_0},q_p}, \tag{22}
$$

where one has to identify $q_0 = k_{a_0} + k_{a_1}, ..., q_p = k_{p_0} + k_{p_1}$. $\mathcal{A}$ is obtained as the Fourier transform in 2-dimensional momentum space of the dipole multiplicity density d_p given in formula (1). In fact, since dipole coordinates correspond to the Fourier transforms of the gluon momenta, $\mathcal{A}$ is to be interpreted as the BFKL $2 \rightarrow 2p$ gluon amplitudes with gluon-gluon singlet channels in the $1/N_c$ limit. Thus the reduced amplitude $A^{h_0,...,h_p}(q_0, ..., q_p)$ is the $1 \rightarrow p$ QCD Pomeron vertex.

Using the expression (18) for $A^{h_0,...,h_p}(q_0, ..., q_p)$, formulae (21,22) have a rather simple and attractive representation (see Fig.1) in terms of a one-loop integral in momentum space with vertices defined by the functions $\mathcal{E}^h_{k,q}$. Each dipole is represented in momentum space and interacts *via* a 3-vertex defined by a function $\mathcal{E}^h_{k,q}$ or $\bar{\mathcal{E}}^h_{k,q}$ depending whether it is created or annihilated. The momentum is conserved at each vertex.

In the next section we will calculate the explicit form of the $\mathcal{E}^h_{k,q}$ vertex functions, which are the building blocks of the obtained expressions, and we will explore their properties.

4 The vertex functions $\mathcal{E}^h_{k,q}$

$\mathcal{E}^h_{k,q}$ is given by the double 2-dimensional Fourier integral (15). By an appropriate sequence of changes of variables, see the appendix **A1**, the expression factorizes to yield

$$
\left[\int d^2w \ e^{iRe(w)} w^{-h} \bar{w}^{-\tilde{h}} \right] \cdot \ \bar{q}^{h-1} q^{\tilde{h}-1} \cdot I_h \left(\frac{\bar{q}}{\bar{k}} \right) \tag{23}
$$

where

$$
I_h(x) = x^{1-h} \bar{x}^{1-\tilde{h}} \int d^2v v^{-h} (1-v)^{-h} (1-xv)^{h-1} \times \{a.h.\} \tag{24}
$$

and the normalization is

$$\left[\int d^2w\ e^{iRe(w)}w^{-h}\bar{w}^{-\tilde{h}}\right] = \frac{\pi i^{h-\tilde{h}}2^{2-h-\tilde{h}}}{\gamma(h)}\ . \tag{25}$$

Here we use the notation $\times\{a.h.\}$ for both complex conjugation and $h \to \tilde{h}$ while $\gamma(h) = \Gamma(h)/\Gamma(1-\tilde{h})$.

The integral in (24) can be performed [13] using the formula quoted in appendix **A1**:

$$I_h(x) = \pi\left[\gamma(1-2h)\gamma^2(h)\ \mathcal{F}_h(x)\mathcal{F}_{\tilde{h}}(\bar{x}) + \{h \to 1-h\}\right] \tag{26}$$

with

$$\mathcal{F}_h(x) \equiv x^h{}_2F_1(h,h;2h|x)\ , \tag{27}$$

where ${}_2F_1$ is the standard hypergeometric function. The final expression for $\mathcal{E}^h_{k,q}$ is therefore

$$\mathcal{E}^h_{k,q} = \frac{\pi i^{h-\tilde{h}}2^{2-h-\tilde{h}}}{\gamma(h)} \cdot \bar{q}^{h-1}q^{\tilde{h}-1} \cdot I_h\left(\frac{\bar{q}}{\bar{k}}\right). \tag{28}$$

Using the resulting formulae (26-28), one obtains an explicit analytical expression for the QCD Pomeron amplitudes. Since the vertex functions appear as their building blocks, let us quote a few mathematical properties of the $\mathcal{E}^h_{k,q}$.

i) $\mathcal{E}^h_{k,q}$ is an $SL(2,\mathbb{C})$ matrix element:

Using formula (45) of the appendix **A2**, one obtains a correspondance with the formula (28) by identifying $-1/(bc)$ with $\bar{q}/\bar{k}$. A simple factorized decomposition of the related group element g can be found:

$$g = \underbrace{\begin{pmatrix} 1 & 0 \\ \bar{k} & 1 \end{pmatrix}}_{\mathfrak{T}_{\bar{k}}} \cdot \underbrace{\begin{pmatrix} 1 & -\bar{q}^{-1} \\ 0 & 1 \end{pmatrix}}_{\mathfrak{h}_{\bar{q}}} \tag{29}$$

Here the first piece $\mathfrak{T}_{\bar{k}}$ is just a translation (*in momentum space*) by $\bar{k}$, while the second element $\mathfrak{h}_{\bar{q}}$ is the $SL(2,\mathbb{C})$ transformation $(0,\infty) \to (0,-\bar{q})$.

ii) Geometrical picture:

Another interesting formula[2] is the following

$$I_h(x) = \int d^2k_\delta \; \bar{E}^h(k_{i\delta}, k_{j\delta}) E^h(k_{i'\delta}, k_{j'\delta}) \tag{30}$$

where $x = \frac{k_{ij}k_{i'j'}}{k_{ii'}k_{jj'}}$ is the anharmonic ratio. Note that one uses the same $SL(2,\mathbb{C})$ Eigenvectors E^h as in (3), but corresponding to $SL(2,\mathbb{C})$ transformations in momentum space. Let us now set $k_i = 0$, $k_j = \infty$, $k_{i'} = \bar{k}$, $k_{j'} = \bar{k} - \bar{q}$. We again recover $I_h\left(\frac{\bar{q}}{\bar{k}}\right)$ as in (23). In fact the $SL(2,\mathbb{C})$ element g appearing in (29) has a simple interpretation as an $SL(2,\mathbb{C})$ element which transforms the momenta $(0,\infty)$ into $(\bar{k}, \bar{k}-\bar{q})$. The origin of the above mentioned factorization (29) is now quite clear in terms of successive geometric transformations in momentum space:

$$(0,\infty) \longrightarrow (0,-\bar{q}) \longrightarrow (\bar{k}, \bar{k}-\bar{q}) \tag{31}$$

By extension, the sequence of group elements corresponding to the product $\mathcal{E}^{h_0}_{k,q_0} \ldots \mathcal{E}^{h_p}_{k-q_0-\ldots-q_{p-1},q_p}$, see (18), corresponds to $SL(2,\mathbb{C})$ group transformations acting in momentum space, and transforming the line $(0,\infty)$ into the $p+1$ lines (forming a polygon) related to the loop of momenta shown in Fig.1. The integration over k (the translation group in momentum space) corresponds to moving the polygon all over the complex plane thus gives an intrinsic geometrical character to the QCD Pomeron correlation functions (18) and amplitudes (21) in momentum space.

5 Similarities with conformal field theories

In the previous sections, we have constructed a solution of the $1 \to p$ QCD Pomeron amplitudes as correlation functions of suitably defined operators acting in a specific Hilbert space. Since our derivation has been motivated by conformal field theory constructions, it is thus natural to discuss the possible connections with conformal field theories

Let us first discuss the conjecture of [7] that dipole correlation functions could be interpreted as correlation functions of quasi-primary operators [9] in some conformal field theory. For instance, in [7] it is found that the 4-point correlation function, which happens to be identical to the $1 \to p, p = 3$

[2] The formula has been originally obtained as a solution for the dipole-dipole Green's function in the impact parameter space [14].

expression (2), can be decomposed into a sum of conformal blocks reminiscent of CFT:

$$B_{1\to 3} \equiv \left\langle 0|\Phi^{h_1}(z_\alpha)\Phi^{h_2}(z_\beta)\Phi^{h_3}(z_\gamma)\Phi^{h_4}(z_\delta)|0\right\rangle = \sum_h C_{h_1 h_2 h} C_{h h_3 h_4}\ |\mathcal{F}(x)|^2, \tag{32}$$

where $x = \frac{\rho_{\alpha\beta}\rho_{\gamma\delta}}{\rho_{\alpha\gamma}\rho_{\beta\delta}}$, C are structure constants and

$$\mathcal{F}(x) = x^{h-h_3-h_4}{}_2F_1(h_3-h_4+h, h_2-h_1+h; 2h|x)\ . \tag{33}$$

These expressions obey CFT crossing relations [7].

However, it turns out that the relation (33) gives stringent constraints on the possible CFT interpretations. In particular, they restrict these CFT to be considered in the classical limit. Indeed, in general CFT the form of the conformal block follows from conformal invariance and depends only on $h_1, \ldots, h_4, h$ and the central charge c. A general closed form expression is unknown but the power series coefficients of $\mathcal{F}^{CFT}(x) = x^{h-h_3-h_4}(1 + \mathcal{F}_1^{CFT}x + \mathcal{F}_2^{CFT}x^2 + \ldots)$ can be calculated order by order [9, 15]. Setting for example $h_1 = h_2$ and $h_3 = h_4$, one has [15]

$$\begin{aligned}\mathcal{F}_2^{CFT} &= \frac{h(h+1)h(h+1)}{4h(2h+1)} + \\ &+2\left(h_1 + \frac{h(h-1)}{2(2h-1)}\right)^2\left(h_3 + \frac{h(h-1)}{2(2h-1)}\right)^2\left(c + \frac{2h(8h-5)}{2h+1}\right)^{-1}\end{aligned} \tag{34}$$

which is similar to the corresponding coefficient $\mathcal{F}_2$ of (33) only if $c = \infty$, that is in the classical limit. If not in this limit, an explicit dependence on h_1 and h_3 remains.

Another constraint on the possible theories is provided by the overcompleteness relation [3] between $SL(2,\mathbb{C})$ Eigenvectors:

$$E^{1-h}(\rho_{10}, \rho_{20}) \propto \int d^2\rho_0'\ \rho_{00'}^{2h-2}\bar{\rho}_{00'}^{2\tilde{h}-2}\ E^h(\rho_{10'}, \rho_{20'}), \tag{35}$$

which would mean in terms of operators that fields labeled by h and $1-h$ can mix. This would be forbidden by conformal invariance, where a linear relation between fields with different conformal weights cannot exist, since vacuum expectation values of a product of operators with different dimension would be non-zero.

This problem can be circumvented by considering WZNW models [15] with $SL(2,\mathbb{C})$ current algebra at some level k. Then one could interpret h as labelling representations of the current algebra and not the conformal dimensions. In this case fields labelled by h and $1-h$ have the same conformal dimension. In particular, in the classical limit[3] we are led to consider $k=\infty$, all fields have dimension 0 and all dependence on the "world-sheet" coordinates vanishes. The coordinates of the fields in the correlation functions are now to be interpreted as auxillary variables [16] labelling states within representations of $SL(2,\mathbb{C})$.

Finally, it is interesting to compare our results with the properties of the $SL(2,\mathbb{C})/SU(2)$-WZNW conformal field theory [12]. It is to be noticed that the functional form of the conformal blocks with auxillary variables [12] become exactly equal to (33), as expected from the minisuperspace limit. Let us now compare the action of primary operators in our case with Ref.[12]. While the dipole operators act by an integral form (8) in the impact-parameter space, they act by multiplication by the vertex function $\mathcal{E}^h_{k,q}$ in momentum space (17), in a similar way as for $SL(2,\mathbb{C})/SU(2)$. However, in formula (17), there is in addition a shift in momentum in the argument of the test function. We know that this shift is essential to obey momentum conservation at each vertex. It will thus be important to identify which conformal field theory structure it would correspond to.

In any case, a possible CFT interpretation of the dipole correlation functions should be compatible with the structure of an effective field theory at one-loop level revealed by our analysis.

6 Conclusions

Let us summarize the results of our study:

i) Using a Hilbert space construction, we were able to formulate the $1\rightarrow p$ dipole multiplicity densities as correlation functions of operators.

ii) Going from impact parameter to momentum space, the correlation functions can be expressed as 1-loop integrals involving universal momentum dependent vertex functions $\mathcal{E}^h_{k,q}$.

iii) The resulting amplitudes correspond to the $1 \rightarrow p$ QCD Pomeron vertices obtained in the $1/N_c$ limit through the dipole formulation.

[3]We do not specify here the explicit form of this model, e.g. the possible coset structure.

iv) The vertex functions are explicitly given in analytic form and possess interesting algebraic and geometrical properties under global conformal $SL(2,\mathbb{C})$ transformations.

These properties are very suggestive of an underlying effective 2 dimensional field theory in the transverse momentum space. We analyzed the conditions that need to be fulfilled in order to have a realization in terms of a local conformal field theory. They imply a minisuperspace (classical) limit and possibly a WZNW symmetry structure.

An important check of these properties would be obtained by a direct calculation of the $2 \to 2p$ gluon amplitudes. Projecting on singlet gluon-gluon channels and taking the $1/N_c$ limit would lead to a direct comparison with our formulae (18) and (22). This would provide a check of a generalized dipole-BFKL correspondence.

Further consequences of these properties can be discussed both from theoretical and phenomenological points of view. On the theoretical ground, the symmetrical formulation of the correlations functions (5,18) with respect to the initial and final dipole fields seems to lead to a relation (at least for $N_c \to \infty$) between $1 \to p+r$ and $p+1 \to r$ dipole multiplicity densities. This intringuing conjecture would deserve further study for instance in the case where $p+1 = r$ which would correspond to a subset of the $2r \to 2r$ gluon amplitudes described in the BKP framework [17].

On the phenomenological ground, the simple formulation of dipole splitting in terms of the $\mathcal{E}^h_{k,q}$ vertices could be useful to describe the evolution of the dipoles through the corresponding amplitudes in momentum space. In particular, it could give a QCD-theoretical hint for the formulation of the formation and splitting of the "QCD string" which is the basis of phenomenological models of strong interactions at long distance.

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A 1: Calculation of the vertex $\mathcal{E}^h_{k,q}$

$$\mathcal{E}^h_{k,q} \equiv \int d^2\rho d^2\rho' \; e^{iq\rho'+ik(\rho-\rho')} \; \frac{1}{|\rho-\rho'|^2} \; \left(\frac{\rho-\rho'}{\rho\rho'}\right)^h \times \{a.h.\} \; . \tag{36}$$

Changing successively $\rho' \to u\rho$ then $w = \rho\left(\bar{k} + u(\bar{q} - \bar{k})\right)$, one obtains a factorized form

$$\left[\int d^2w \; e^{iRe(w)} w^{-h} \bar{w}^{-\tilde{h}}\right] \cdot \; \bar{k}^{h-1} k^{\tilde{h}-1} \int d^2u u^{-h} (1-u)^{h-1} (1+(x-1)u)^h \times \{a.h.\} \; , \tag{37}$$

where $x = \frac{\bar{q}}{\bar{k}}$. Finally the change of variable $v = \frac{u}{u-1}$ in the second integral leads to the formula (24) for $I_h(x)$.

This integral is known in the literature [13]. Consider the following more general form:

$$\mathcal{I}(a_0, a_1, b_1, z) \equiv \int d^2v \; v^{a_1-1} (1-v)^{b_1-a_1-1} (1-vz)^{-a_0} \times \{a.h.\} \tag{38}$$

The result is [13]:

$$\mathcal{I}(a_0, a_1, b_1, z) = \pi \Big[\frac{\gamma(a_1)\gamma(b_1-a_1)}{\gamma(b_1)} {}_2F_1(a_0, a_1; b_1|z) \times \{a.h.\} +$$

$$+ \frac{\gamma(b_1-1)\gamma(1-a_0)}{\gamma(b_1-a_0)} z^{1-b_1} {}_2F_1(a_0-b_1+1, a_1-b_1+1; 2-b_1|z) \times \{a.h.\} \Big] \; , \tag{39}$$

where we use the notation $\times\{a.h.\}$ for both complex conjugation and $h \to \tilde{h}$.

A 2: $SL(2,\mathbb{C})$ matrix elements

Consider the representation of $SL(2,\mathbb{C})$ given by

$$[T(g)f](z) = (cz+d)^{-2h} \times \{a.h.\} \cdot f\left(\frac{az+b}{cz+d}\right) \tag{40}$$

where

$$g = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \; , \tag{41}$$

and the set of orthogonal basis functions

$$|\omega,\tilde{\omega}\rangle = z^{-h-\omega}\bar{z}^{-\tilde{h}-\tilde{\omega}} \tag{42}$$

with $\omega = m/2 - i\sigma$, $\tilde{\omega} = -m/2 - i\sigma$ where $m \in \mathbb{N}$ and $\sigma \in \mathbb{R}$. These functions are eigenfunctions of the (complex) dilatation transformations:

$$T\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right)|\omega,\tilde{\omega}\rangle = \lambda^{-2\omega}\bar{\lambda}^{-2\tilde{\omega}}\,|\omega,\tilde{\omega}\rangle \tag{43}$$

In particular the only dilatationally invariant state is $|0,0\rangle$. The matrix element of $T(g)$ in this basis can be evaluated to yield

$$\begin{aligned}\langle\omega_L,\tilde{\omega}_L|T(g)|\omega_R,\tilde{\omega}_R\rangle = a^{\tilde{\omega}_L^*-\omega_R}(-b)^{h-1-\tilde{\omega}_L^*}(-c)^{h-1+\omega_R} \times \{a.h.\}\,\cdot \\ \cdot\int d^2v\; v^{-h-\omega_R}(1-v)^{-h+\omega_R}(1+\frac{v}{bc})^{h-1-\tilde{\omega}_L^*} \times \{a.h.\}\;.\end{aligned} \tag{44}$$

For $\omega_L = \tilde{\omega}_L = \omega_R = \tilde{\omega}_R = 0$ we get exactly

$$\langle 0,0|T(g)|0,0\rangle = I_h\left(\frac{-1}{bc}\right)\;. \tag{45}$$

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